

DOCUMENT RESUME

ED 349 179

SE 053 185

TITLE Guidelines for Teaching Calculus.
INSTITUTION Texas Education Agency, Austin. Div. of Curriculum Development.
REPORT NO GEO-621-03
PUB DATE 90
NOTE 56p.
AVAILABLE FROM Publications Distributions Office, Texas Education Agency, 1701 North Congress Avenue, Austin, TX 78701 (\$1.00).
PUB TYPE Guides - Classroom Use - Teaching Guides (For Teacher) (052) -- Tests/Evaluation Instruments (160)

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS Calculators; *Calculus; College Mathematics; Computers; *Course Objectives; High Schools; *Learning Activities; Mathematical Concepts; Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; *Mathematics Tests; Secondary School Mathematics; Teaching Guides; Teaching Methods
IDENTIFIERS Limits (Mathematics); *Texas Education Agency

ABSTRACT

This course guide for calculus offered at the high school level begins with an overview of the course. The overview includes a rationale for including calculus in the high school curriculum, a description and rationale of calculator and computer use in the calculus class, the desired qualifications and characteristics of calculus teachers, prerequisites for students entering calculus, and the basis on which credits are awarded to calculus students. The guide is divided into three sections. The first section describes the essential elements of the calculus course based on the Advanced Placement Calculus AB course as approved by the State Board of Education, a list of the course objectives for the first and second semesters, and a series of 10 sample activities for the teacher. The activities each identify the topic covered, the sample objective to be accomplished, background information for the activity, the instructional approach employed, and the technology utilized. The second section contains nine sample tests for various calculus topics that can be adapted for the specific needs determined by the teacher. The third section provides the following resource lists: advanced placement material available from the Advanced Placement Program; State-Adopted Textbooks; supplementary textbooks; reference materials; software; and professional associations providing information, materials and/or conferences related to the teaching of mathematics. (MDH)

* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *

ED340179

Guidelines for Teaching Calculus

JE053185

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

- ☒ This document has been reproduced as
received from the person or organization
originating it.
- ☐ Minor changes have been made to improve
reproduction quality.

- Points of view or opinions stated in this docu-
ment do not necessarily represent official
OERI position or policy.

Texas Education Agency
Austin, Texas

BEST COPY AVAILABLE

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY

Linda Kemp

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

Guidelines for Teaching Calculus

This publication is not copyrighted; any or all sections may be duplicated. After an initial free distribution to authorized institutions, additional copies may be purchased for \$1.00 from the Publications Distributions Office.

**Texas Education Agency
1701 North Congress Avenue
Austin, Texas 78701**

FOREWORD

Guidelines for Teaching Calculus is designed to assist teachers and other school district personnel in the planning and teaching of the high school course Calculus. The publication presents the philosophy and intent of the course and discusses the use of technology, teacher preparation, credits and prerequisites, and the required essential elements of instruction. Also included are sample objectives, sample activities, sample tests, and resource materials. The sample objectives, activities, and tests illustrate how the essential elements for Calculus can be taught. School district personnel may use these suggestions to develop their own curriculum documents for Calculus.

We hope these guidelines will be useful to district personnel in planning curriculum, equipping classrooms, and teaching Calculus.

W.N. Kirby
Commissioner of Education

STATE BOARD OF EDUCATION

(State Board for Vocational Education)

MONTE HASIE, Lubbock
Chairman of the State Board of Education
District 15

CAROLYN HONEA CRAWFORD, Beaumont
Vice Chairman of the State Board of Education
District 7

MARY HELEN BERLANGA, Corpus Christi
Secretary of the State Board of Education
District 2

Board Members

BOB AIKIN, Commerce
District 9

RAY ALEXANDER, Houston
District 4

MARY HELEN BERLANGA, Corpus Christi
District 2

EMMETT J. CONRAD, Dallas
District 13

CAROLYN HONEA CRAWFORD, Beaumont
District 7

ROBERT CUMMINGS, Houston
District 6

WILL DAVIS, Austin
District 10

WILLIAM L. HUDSON, Wichita Falls
District 14

GERALDINE MILLER, Dallas
District 12

JANE NELSON, Lewisville
District 11

RENE NUÑEZ, El Paso
District 1

MARY PERKINS, Lufkin
District 8

JOHN H. SHIELDS, San Antonio
District 5

ESTEBAN SOSA, San Antonio
District 3

W.N. KIRBY, Commissioner of Education
(Executive Officer of the State Board of Education)

**Committees
of the
State Board of Education**

PERMANENT SCHOOL FUND

ESTEBAN SOSA, Chairman
BOB AIKIN
MONTE HASIE
WILLIAM L. HUDSON
GERALDINE MILLER
RENE NUÑEZ

SCHOOL FINANCE

WILL D. DAVIS, Chairman
BOB AIKIN
ROBERT D. CUMMINGS
WILLIAM L. HUDSON
ESTEBAN SOSA

PERSONNEL

CAROLYN HONEA CRAWFORD, Chairman
EMMETT J. CONRAD
RENE NUÑEZ
JOHN H. SHIELDS

LONG-RANGE PLANNING

EMMETT J. CONRAD, Chairman
ROBERT D. CUMMINGS
WILL D. DAVIS
JANE NELSON
MARY KNOTTS PERKINS
JOHN H. SHIELDS

STUDENTS

GERALDINE MILLER, Chairman
RAYMOND A. ALEXANDER
MARY HELEN BERLANGA
MONTE HASIE
JANE NELSON
MARY KNOTTS PERKINS

TEXAS EDUCATION AGENCY

This publication was developed by Agency staff members in the Office of the Deputy Commissioner for Curriculum and Professional Development.

VICTORIA BERGIN

Deputy Commissioner for Curriculum and Professional Development

GRACE GRIMES

Assistant Commissioner for Curriculum and Instruction

LEROY F. PSENCIK

Director, Division of General Education

Mathematics Section

CATHY SEELEY

Director of Mathematics

BILLY L. HOPKINS

Mathematics Specialist, High School

REBECCA BROOKS McCOY

Mathematics Specialist, Elementary

BARBARA MONTALTO

Mathematics Specialist, Middle School
and Junior High

PAT FOLMAR ROBINSON

Editor, Division of General Education

CONTENTS

Overview of the Course	1
Rationale	1
The Use of Technology	1
Teacher Preparation	2
Prerequisites and Credits	3
 Essential Elements of Instruction, Objectives, and Activities	 4
Essential Elements	4
Sample Objectives	6
Sample Activities	11
 Evaluation: Sample Tests	 33
 Resources	 42
Advanced Placement Examinations	42
State-Adopted Textbooks	42
Supplementary Textbooks	43
Reference Materials	43
Software	44
Associations	45
 Acknowledgments	 46

Overview of the Course

Rationale

An understanding of the principles of calculus is essential for many fields of study, including mathematics, the sciences, engineering, economics, sociology and psychology, statistics, computer programming, and system analysis. The study of calculus is also inherently valuable. For example, calculus requires the use of all of a student's mathematical knowledge; thus, calculus students begin to see mathematics as a whole rather than just as individual segments. Also, calculus students continuously use problem-solving skills, the development of which is a major focus of current mathematics programs across the nation.

Calculus is a logical next course for many high school students who have completed four years of college preparatory mathematics and who have demonstrated mastery of algebra, geometry (including coordinate geometry), and trigonometry. The high school calculus course should be as rigorous as courses taught at the college level. Some high schools offer the equivalent of a one-semester college course while others provide the equivalent of a full-year college course. In both cases, the course should provide an understanding of the essential principles of calculus.

The Use of Technology

Many mathematics educators support effective use of scientific calculators and microcomputers in mathematics instruction. As early as 1980, the National Council of Teachers of Mathematics (NCTM) recommended that "mathematics programs take full advantage of the power of calculators and computers at all grade levels. . . . It is imperative that schools play an active part in preparing students . . . to live in a world in which more and more functions are being performed by computers."¹ Similarly, in 1983 the National Science Board Commission reported: "The use of computers and other high technology equipment in the classroom should be managed so that students receive both the benefits of the equipment and increased familiarity with this technology."²

¹National Council of Teachers of Mathematics, *An Agenda for Action* (Reston, VA: NCTM, 1980).

²National Science Board Commission on Precollege Education in Mathematics, Science, and Technology, *Educating Americans for the 21st Century* (The Commission, 1983).

In *Curriculum and Evaluation Standards for School Mathematics*, NCTM notes that "the (calculus) instructional approach . . . should be highly exploratory and based on geometric and numerical experiences that capitalize on computer and calculator-based graphing utilities. The instructional activities should be aimed at providing students with firm conceptual underpinnings of calculus rather than with the associated manipulative techniques."³

Calculators can be particularly useful to students studying calculus. Teachers can assign a greater number of problems, using more realistic real-world data, without overburdening students. Students can then devote time to understanding concepts rather than cumbersome computations. For example, use of calculators greatly enhances students' understanding of the concept of limit and numerical approximations by the trapezoidal and Simpson's rules.

Graphing calculators provide additional options for calculus students. Students can build functions with symbolic arithmetic, then take derivatives symbolically, and finally display graphs of the functions and derivatives. As an extension, students can use graphs of two functions on the same set of axes for the graphical calculation of their intersection. Additionally, a graphing calculator permits study of points of inflection, maximal and minimal values, and changes in curvature. Another application is to construct a function's Taylor Series to a specified degree and then compare the graphs of the function and the Taylor Series, thus visualizing the geometrical separation of the two curves.

Computers can be used in several areas of calculus instruction: in demonstrations, for problem solving, and for student discovery. Software packages that plot graphs of user-designated functions and relations allow students to visualize properties readily. The computer's power and speed permits a larger amount of exploration than is possible manually. Increasingly, textbook publishers are providing software for instructional units in which computer utilization enhances understanding.

To help take advantage of computers and calculators as important tools in instruction, teachers would benefit from having computers with LCD overhead projection devices for demonstration purposes, as well as overhead projection graphing calculators. In addition, calculus teachers should use a diverse selection of other instructional resources, including well-planned use of media such as overhead projectors, videotapes, films, slides, and television. One example of a beneficial film is "Stand and Deliver," which depicts the true story of a class of students in Los Angeles that, in less than two years of study, progressed from basic mathematics through calculus. Eventually, the class achieved 100 percent success on the Advanced Placement test. Likewise, a number of good calculus-related transparencies, some designed as supplementary aids to texts, are available. Effective use of these and other current technological resources help students to develop fully their mathematical knowledge and power.

Teacher Preparation

Calculus must be taught by a teacher who is certified in secondary mathematics. A teacher should have a strong background in mathematics including calculus and upper division mathematics courses. A master's

³National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics* (Reston, VA: NCTM, 1989).

degree with a major in mathematics is desirable but not required. Although courses beyond Calculus are not generally offered in high school, a teacher with a background in upper division mathematics courses will be better prepared to teach applications and enrichment topics in Calculus.

High School calculus teachers should not only have superb academic backgrounds in mathematics but should also be current with respect to developments in the teaching of mathematics and technological advances that impact teaching. They should be dedicated professionals who attend mathematics conferences and who read professional journals to stay abreast of developments in their field. They should never stop learning and should always look for new and better ways to teach the topics in their courses. In particular, they should be acutely aware of the technological advances that can facilitate and enhance students' understanding of mathematics, and they should be committed to the routine and regular use of both hand-held calculators and computers in calculus instruction. Finally, teachers of the high school Calculus course should love and respect the power and beauty of mathematics and should be dedicated to sharing that love and respect as well as their knowledge of calculus with their students.

Prerequisites and Credits

Students entering Calculus should have successfully completed either Precalculus or Trigonometry and either Elementary Analysis or Analytic Geometry. Students may receive one-half unit of credit for successfully completing one semester of Calculus or one unit of credit for successfully completing two semesters of Calculus.

Essential Elements of Instruction, Objectives, and Activities

Essential Elements

The following list of revised essential elements for Calculus have been approved by the State Board of Education. These essential elements basically outline the Advanced Placement Calculus AB course, which is equivalent to a typical one-semester college calculus course. These essential elements became effective Fall 1989.

1. Concepts and skills associated with elementary functions. The student shall be provided opportunities to:
 - (A) use algebraic, trigonometric, exponential, and logarithmic functions;
 - (B) investigate properties of functions including:
 - (i) sum, product, quotient, composition, and inverse (including domain and range of each);
 - (ii) absolute value;
 - (iii) odd and even; and
 - (iv) zeros of a function.
 - (C) use fundamental identities including addition formulas for trigonometric functions.
 - (D) graph functions using:
 - (i) symmetry;
 - (ii) asymptotes; and
 - (iii) periodicity and amplitude.
2. Concepts associated with the limit of a function. The student shall be provided opportunities to:
 - (A) apply limit theorems and properties;
 - (B) investigate special limits, including:
 - (i) the limit defining the Euler number e ;
 - (ii) limits involving trigonometric functions; and
 - (iii) nonexistent limits.

- (C) use continuity, including the Intermediate Value Theorem.
3. Concepts and skills associated with the derivative. The student shall be provided opportunities to:
- (A) find tangent and normal lines to a curve;
 - (B) derive formulas for the derivatives from the limit of a function;
 - (C) use theorems and properties of the derivative including the relationship between differentiability and continuity;
 - (D) apply the Mean Value Theorem; and
 - (E) sketch curves, using
 - (i) techniques for identifying intervals on which the curve is increasing or decreasing;
 - (ii) relative and absolute maximum and minimum points;
 - (iii) concavity; and
 - (iv) points of inflection.
 - (F) apply concepts and skills including
 - (i) velocity and acceleration;
 - (ii) related rates;
 - (iii) maxima and minima; and
 - (iv) l' Hôpital's rule.
 - (G) differentiate special functions such as
 - (i) trigonometric functions;
 - (ii) logarithmic functions; and
 - (iii) exponential functions.
4. Concepts and skills associated with the integral and techniques of integration. The student shall be provided opportunities to:
- (A) find antiderivatives (indefinite integrals) of functions;
 - (B) apply antiderivatives to solve problems including:
 - (i) distance and velocity from acceleration with initial conditions; and
 - (ii) differential equations with separable variables including growth and decay.
 - (C) investigate the definite integral including:
 - (i) the concept of the definite integral as an area;
 - (ii) approximations to the definite integral using rectangles;
 - (iii) definition of the definite integral as the limit of a sum;
 - (iv) properties of the definite integral; and
 - (v) the fundamental theorems of integral calculus.

- (D) apply concepts of the definite integral to solve problems involving:
 - (i) the average value of a function;
 - (ii) area between curves;
 - (iii) volume of a solid of revolution (disc, washer, and shell methods); and
 - (iv) length of a curve.
- (E) use techniques of integration including:
 - (i) basic integration formulas;
 - (ii) integration by substitution;
 - (iii) integration by parts; and
 - (iv) integration by partial fractions.
- (F) integrate special functions including:
 - (i) trigonometric functions;
 - (ii) logarithmic functions; and
 - (iii) exponential functions.

5. Applications of calculus to special functions.

Sample Objectives

District educators should develop or adapt objectives that will ensure that all the required essential elements of Calculus are taught. The objectives should clarify the essential elements (listed on pages 4-6) and describe expected student outcomes.

A number of sample objectives for Calculus are listed below. District curriculum planners may use them in the development of their own course objectives. The objectives are arranged in sequence by semester. Correlations to the essential elements follow each objective. Also, noted to the left of many of the objectives are symbols that indicate the objective is appropriate for using a computer (Com), a calculator (Cal), or a graphic calculator (GC). Although certain objectives are especially appropriate for using calculators or computers, the teaching and learning of virtually every objective can be enhanced through the use of technology.

The essential elements for Calculus are designed to address the objectives of Advanced Placement Calculus AB. However, many Calculus teachers may be able to include more than the AB curriculum. Additional sample objectives for the Advanced Placement Calculus BC curriculum are included below. These objectives are marked with a star (☆). Teachers are encouraged to teach as many of the BC objectives as possible.

Objectives 59-84 relating to hyperbolic functions are not a part of Advanced Placement Calculus; however, most college courses include them. The hyperbolic function objectives would be helpful to students in a BC course.

First Semester Objectives

☆ See page 6

The student will:

	1. find the domain and range of a function [EE 1(A), 1(B)]
	2. combine two functions, using algebraic operations [EE 1(B)]
	3. determine if a function is odd or even [EE 1(B)]
Com, GC	4. find the inverse of a function both algebraically and graphically [EE 1(B)]
Com, GC	5. graph functions using symmetry and asymptotes. [EE 1(D)]
Com, GC	6. graph variations of (a) the absolute value function, (b) the greatest integer function, and (c) the signum function [EE 1(B)]
Com	7. calculate the slope of a curve at a point $P(x, y)$ as the slope of the tangent line to the curve at P [EE 3(A)]
	8. develop two forms of the definition of the derivative [EE 3(B)]
	9. find the derivative of a function $f(x)$ by the definition [EE 3(B)]
	☆10. find both average velocity and instantaneous velocity in meaningful situations [EE 3(F)]
	☆11. apply the epsilon-delta definition of a limit to find delta for a given epsilon [EE 2(A)]
Cal	12. apply the limit theorems to evaluate the limit of a function $f(x)$ as x approaches a finite value a [EE 2(A)]
	13. apply the formulas for the derivatives of (a) sum, (b) product, (c) quotient, and (d) power of differentiable functions [EE 3(B)]
	14. find velocity and acceleration, given the position function for a particle moving along a line [EE 3(F)]
	15. apply implicit differentiation to relations not stated explicitly as functions of x [EE 3(B)]
Cal, Com	16. apply Newton's method of approximating roots of polynomial functions [EE 1(B)]
	17. find the composition of two functions and apply the chain rule to find the derivative [EE 3(C), 3(G)]

First Semester Objectives (continued)

☆ See page 6.

Com, GC	18. calculate the amplitude and period of trigonometric functions and draw their graphs [EE 1(D)]
	19. apply the theorem $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ to evaluate trigonometric limits [EE 2(A), 2(B)]
	20. find the derivative of sines and cosines [EE 3(G)]
	21. determine if a function is continuous at a point $P(x_0, y_0)$ by the definition [EE 2(C)]
	22. determine if a function is continuous at a point $P(x_0, y_0)$ by applying knowledge of the differentiability of the function at that point [EE 3(C)]
Cal	23. approximate a value of a function by using differentials
	24. write equations of tangents and normals to a curve at a point $P(x_0, y_0)$ [EE 3(A)]
	25. sketch curves by applying the first and second derivatives to find relative maxima and minima, points of inflection, and concavity [EE 3(E)]
	26. solve related rates problems [EE 3(F)]
	27. solve problems involving maxima and minima [EE 3(F)]
	28. apply the Mean Value Theorem to find a point where the tangent to a curve $f(x)$ is parallel to the chord AB [EE 3(D)]
	29. use l' Hôpital's Rule to evaluate appropriate indeterminate limits [EE 3(F)]
	30. find antiderivatives (indefinite integrals), given the derivative function [EE 4(A)]
	31. solve simple differential equations by separating the variables and finding the indefinite integrals [EE 4(B)]
Cal, Com	32. find approximations and actual area under a curve by: (a) rectangles ☆ (b) trapezoids ☆ (c) Simpson's Rule (d) Fundamental Theorem of Calculus [EE 4(C)]
	☆33. find $F'(x)$ if $F(x) = \int_a^{u(x)} f(t)dt$ [EE 4(C)]

Second Semester Objectives

☆ See page 6

The student will:

Com	34. find the area between two curves [EE 4(D)]
Cal	35. find the total and/or net distance a particle has traveled when given the velocity and/or acceleration [EE 4(B)]
Cal, Com	36. find the volume of a solid formed by rotating a given region about a given line by (a) the disc method and (b) the shell method [EE 4(D)]
Com, Cal	37. find the volume of a solid of revolution with a known cross section
	☆38. find the length of a rectifiable curve [EE 4(D)]
	☆39. find the area of a surface of revolution
	40. find the average (mean) value of a function over a closed interval [EE 4(D)]
	41. differentiate and integrate trigonometric functions [EE 3(G), 4(F)]
	42. differentiate inverse trigonometric functions [EE 3(G)]
	43. integrate the derivatives of inverse trigonometric functions [EE 4(F)]
Cal	44. define e as a limit [EE 2(B)]
	45. perform operations with the natural log function and the exponential function [EE 1(A), 1(B)]
	46. differentiate and integrate functions involving $\ln u$ and e^u [EE 3(G), 4(F)]
	47. find derivatives and integrals involving a^u and $\log_a u$ [EE 3(G), 4(F)]
	48. apply the technique of logarithmic differentiation
	49. solve simple differential equations involving $\ln u$ and e^u [EE 4(B)]
	50. integrate powers of trigonometric functions [EE 4(F)]
	51. use the technique of integration by substitution [EE 4(E)]
	☆52. integrate functions by trigonometric substitution, trigonometric integrals [EE 4(E)]
	☆53. integrate functions by the method of partial fractions [EE 4(E)]

Second Semester Objectives (continued)

☆ See page 6

	54. perform simple integration by parts [EE 4(E)]
	☆55. do more complex (repeated) integration by parts [EE 5]
	☆56. evaluate improper integrals or show that they diverge
Cal, GC	57. write equations of conics and draw their graphs
Cal	58. use a translation or rotation of axes to simplify the equation of a conic
Cal	☆59. evaluate hyperbolic functions
	☆60. differentiate hyperbolic functions [EE 3(G)]
	☆61. integrate hyperbolic functions [EE 4(F)]
	☆62. differentiate inverse hyperbolic functions [EE 3(G)]
	☆63. find integrals that involve inverse hyperbolic functions [EE 4(F)]
	☆64. solve problems involving the catenary (hanging cable) [EE 4(F), 5]
	☆65. transform polar equations into rectangular and rectangular into polar form
Com, GC	☆66. graph polar equations
	☆67. find the plane area inside a polar curve [EE 4(C)]
	☆68. find the area between two polar curves [EE 4(D)]
	☆69. express a vector as a linear combination of the unit vectors i and j
	☆70. eliminate the parameter in parametric equations to obtain a Cartesian equation
	☆71. find derivatives of parametric equations [EE 3(G)]
	☆72. differentiate a vector function [EE 3(G)]
	☆73. find velocity and acceleration vectors for motion on a plane curve [EE 5]
	☆74. find the speed of a particle on a plane curve
	☆75. determine if a sequence of real numbers (or functions) converge

-
- ☆76. be familiar with geometric series

 - ☆77. determine if a series of positive terms converge by using the comparison, ratio, and integral tests

 - ☆78. determine if a series converges absolutely

 - ☆79. find error approximations of alternating series

 - ☆80. determine if an alternating series converges

 - ☆81. find the Taylor series for e^u , $\sin u$, $\cos u$, and algebraic functions

 - ☆82. evaluate Taylor series with remainders

 - ☆83. find the interval of convergence of a power series

 - ☆84. differentiate and integrate power series [EE 3(G), 4(F)]

 - ☆85. solve differential equations of the first order with variables separable [EE 4(B)]

 - ☆86. solve linear differential equations of the first order with constant coefficients (both homogeneous and nonhomogeneous) by the method of undetermined coefficients

 - ☆87. solve applications with initial conditions [EE 5]

Sample Activities

The following sample activities illustrate how teachers may address different topic areas by having students perform the prescribed tasks:

Activity 1 [Calculator, Step (4)]

Topic

Epsilon-Delta Definition of Limit

Sample Objective 11

Apply the epsilon-delta definition of limit to find delta for a given epsilon.

Background

The epsilon-delta definition of limit is extremely important in calculus because of the critical impact of the limit concept. The definition is unfamiliar to students, but teachers can help by stating, explaining, and using the definition in every college-preparatory calculus class. This activity contains a standard epsilon-delta proof, an exercise that can help students understand the results of the proof, and ideas for allowing students to extend the definition as the course progresses.

Instructional Approach

- (1) State the formal definition of limit as follows:

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if}$$

for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$,
then
 $|f(x) - L| < \epsilon$.

- (2) Use the definition to verify the limit for a specific example. As an example for students, demonstrate the proof process for

$$\lim_{x \rightarrow 2} (4x - 3) = 5 \quad (\text{See Figure 1.})$$

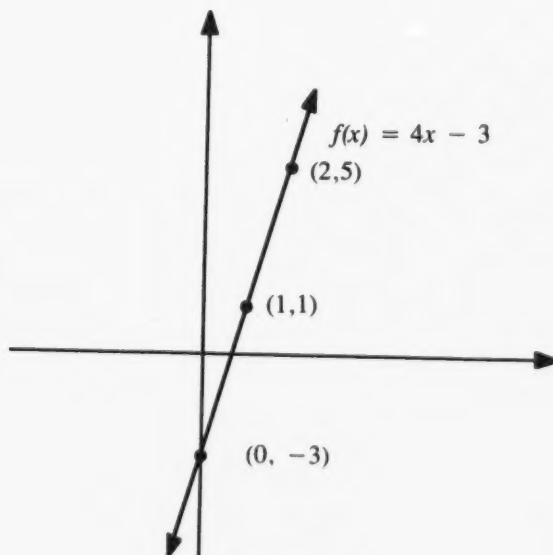


FIGURE 1

Students usually have better success with epsilon-delta proofs if they work backward first. The advantage to working backward is that a suitable delta appears, and the student need only reverse the steps to accomplish the proof.

Working backward:

$$|f(x) - L| < \epsilon$$

$$|(4x - 3) - 5| < \epsilon$$

$$|4x - 8| < \epsilon$$

$$4|x - 2| < \epsilon$$

$$|x - 2| < \frac{\epsilon}{4}$$

- (3) Using the results in (2), write the formal proof as follows:

$$\text{Let } \delta = \frac{\epsilon}{4}$$

$$0 < |x - 2| < \frac{\epsilon}{4}$$

$$0 < |x - 2| \text{ implies } x \neq 2$$

$$|x - 2| < \frac{\epsilon}{4}$$

$$4|x - 2| < \epsilon$$

$$|4x - 8| < \epsilon$$

$$|4x - 3 - 5| < \epsilon$$

$$|(4x - 3) - 5| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$\text{Therefore, } \lim_{x \rightarrow 2} (4x - 3) = 5$$

- (4) Using calculators, have students verify the results above.

Ask a student to select a value for epsilon; it might be a good idea to discuss the nature of epsilon and delta and to agree that epsilon should be between 0 and 1. Suppose the student chooses $\epsilon = 0.01$. Discussion should reinforce the idea that the choice of 0.01 means that values of x must exist which are "close enough" to 2 to ensure that $f(x)$ or $(4x - 3)$ will be **within** 0.01 unit of 5. Now, how close to 2 is "close enough?" Delta provides the answer to that question.

In Step (2), a suitable choice for delta was found to be epsilon divided by 4. Thus, try $\delta = 0.0025$. Now, any x such that x is within 0.0025 of 2 should produce a value for $f(x)$ that is within 0.01 of 5.

Symbolically, if $1.9975 < x < 2.0025$, then $4.99 < 4x - 3 < 5.01$.

In general, if $a - \delta < x < a + \delta$, then $L - \epsilon < f(x) < L + \epsilon$.

Ask a student to select a suitable x . (If a student chooses one of the endpoints of the interval, ask him or her to review the definition of limit. These values are not available for choice.) Suppose a student chooses $x = 2.001$. This is an appropriate choice for x as it is within 0.0025 of 2 and is different from 2.

Calculate $f(2.001)$.

$$\begin{aligned}\text{Since } f(x) &= 4x - 3, f(2.001) = 4(2.001) - 3 \\ &= 8.004 - 3 \\ &= 5.004\end{aligned}$$

Note that 5.004 is within 0.01 of 5!

By choosing an x within delta of 2, we have found an $f(x)$ within epsilon of 5.

NOTE: Students should understand that this activity does not constitute a proof. The proof occurs in Step (3). This is, however, an excellent way to help students understand just what they have proved in Step (3). This exercise also provides a good opportunity for students to communicate mathematics as they talk and write about the activity.

(5) Extend the basic epsilon-delta definition of limit.

Consider the concept of limits at infinity. It may be tempting to deal with this concept using only intuition, but instead, review the formal epsilon-delta definition.

Ask students leading questions such as the following, and give them opportunities to extend the basic definition. This is another opportunity to involve them in writing and talking about mathematics.

- Should there be an epsilon?

Remember that epsilon is a small positive number that relates to how close $f(x)$ is to L .

- Should there be a delta?

Remember that delta is a small positive number that relates to how close x is to a .

- What happens to some familiar functions as x goes to positive or negative infinity?

Examine some familiar functions with a graphing calculator or a computer to recall what is happening to $f(x)$ as x goes to positive or negative infinity.

Look at graphs of the following functions and make conjectures as to the following limits:

$$f(x) = x$$

$$\lim_{x \rightarrow \infty} x =$$

$$f(x) = e^x$$

$$\lim_{x \rightarrow \infty} e^x =$$

$$f(x) = \ln x$$

$$\lim_{x \rightarrow \infty} \ln x =$$

$$f(x) = \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} =$$

$$f(x) = x^3 - 4x^2 + 3x - 2$$

$$\lim_{x \rightarrow \infty} (x^3 - 4x^2 + 3x - 2) =$$

$$f(x) = \arctan x$$

$$\lim_{x \rightarrow \infty} \arctan x =$$

Allow students to discover and to discuss the fact that in all of the examples, x is getting large without limit; thus, the definition must not include a delta. Students should also notice that in some examples $f(x)$ is increasing or decreasing without limit, too, but in others $f(x)$ appears to be bounded. Using calculators allows students to make excellent conjectures as to those bounds in the cases of the arctan function and the reciprocal function above.

- (6) Now adjust the definition to deal with the case in which $f(x)$ is bounded.

$$\lim_{x \rightarrow \infty} f(x) = L \text{ if and only if}$$

for each $\epsilon > 0$ there exists a $N > 0$ such that if $x > N$, then $|f(x) - L| < \epsilon$.

Let students write the definition for the case when $f(x)$ is unbounded, and use this same discovery approach for developing other variations of the epsilon-delta definition.

Activity 2

Topic

The Definition of the Derivative

Sample Objective 8

Develop or discover two forms of the definition of the derivative.

Background

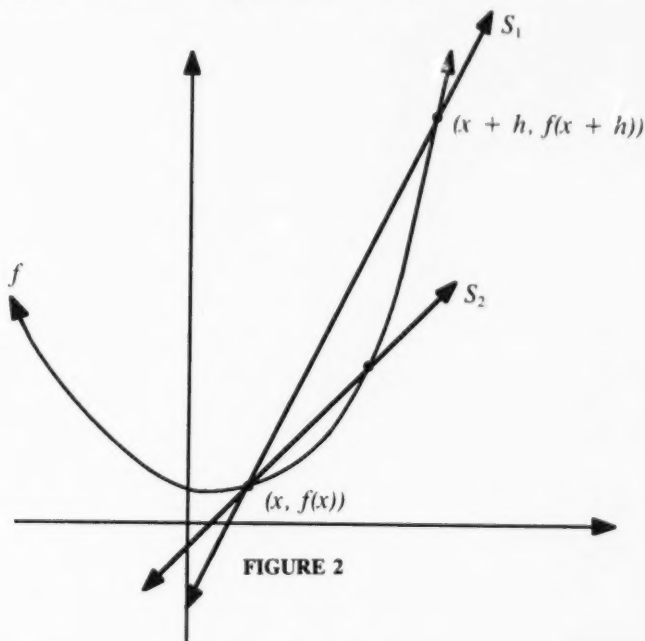
This activity helps students to develop a firm understanding of the definition of the derivative. By observing the slopes of a system of tangent lines, students can discover the definition of the derivative as the slope of the line tangent to the curve at a point.

The first form (or x form) emphasizes points about x that are h units from x . As h approaches 0, the points $x + h$ move closer to x . The second form (or a form) fixes $(a, f(a))$, and the points x approach the fixed point a .

Instructional Approach

First Form

- (1) Given a function f , have students examine the slope for a system of secant lines, each of which contains the fixed point $(x, f(x))$ and the moving point $(x + h, f(x + h))$.



$$\text{For } S_1, m = \frac{f(x+h) - f(x)}{h} \text{ or } m = \frac{f(x+h) - f(x)}{(x+h) - x}$$

For S_2 , choose $(x+h, f(x+h))$ closer to $(x, f(x))$, and note that

$$m = \frac{f(x+h) - f(x)}{h}$$

Use a graphics calculator or sketch additional secant lines in which $(x+h, f(x+h))$ is closer with each successive observation to $(x, f(x))$. (See Figure 2). Sketching should be done with a sharp pencil.

(2) Ask students questions such as:

- Is the expression for slope independent of the position of the point $(x + h, f(x + h))$?
- In determining a member of the secant line system, can $x + h = x$? That is, can the two points coincide?
- What is happening to the number h as the two points get very close together?
- Can a secant line ever be a tangent line? (At this point, students have only an intuitive idea of what a tangent line is.)
- Is there a limiting position for the members of the system of secant lines as the two points get very close together?
- If the secant lines have a limiting position, would the expression for slope have a limit? If so, what would that limit be? (The limit should be the slope of the tangent line at $(x, f(x))$.)

(3) State the x form of the definition of the derivative. The first derivative of the function f , symbolized in one of several ways as $f'(x)$, is defined as follows:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Second Form

(1) Given a function f , examine the slope for a system of secant lines, each of which contains the fixed point $(a, f(a))$ and the moving point $(x, f(x))$.

$$\text{For } S_1, m = \frac{f(x) - f(a)}{x - a}$$

For S_2 , choose $(x, f(x))$ closer to $(a, f(a))$, and note that

$$m = \frac{f(x) - f(a)}{x - a}$$

Use a graphics calculator or sketch additional secant lines in which $(x, f(x))$ is closer with each successive observation to $(a, f(a))$. (See Figure 3 on next page.)

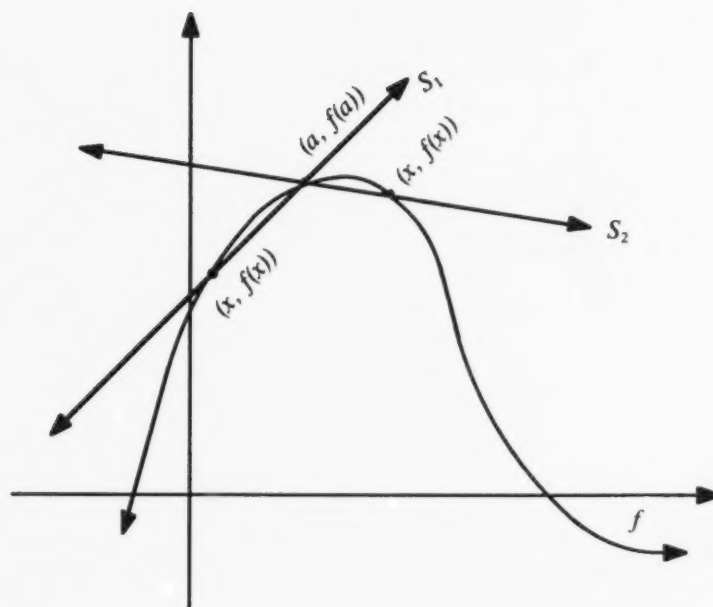


FIGURE 3

- (2) State the a form of the definition of the derivative. The first derivative of the function f , symbolized in one of several ways as $f'(x)$, is defined as follows:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

- (3) Compare the x form of the definition of the derivative with the a form of the definition.

Activity 3

Topic

Implicit Differentiation

Sample Objective 15

Apply implicit differentiation to relations not stated explicitly as functions.

Background

Equations of the form $y = f(x)$ define y **explicitly** as a function of x . Often a relation in x and y cannot be solved for y as a function of x . However, various parts of the graphs of such an equation in x and y may define y as a function of x . Such functions are said to be given **implicitly** by the equation. When these functions are differentiable, their derivatives may be found without solving the equation for y . The method used is called implicit differentiation.

Instructional Approach

- (1) Consider the circle with center at the origin and radius 5 units (see Figure 4.)

Its equation is $x^2 + y^2 = 25$ or $y = \pm \sqrt{25 - x^2}$.

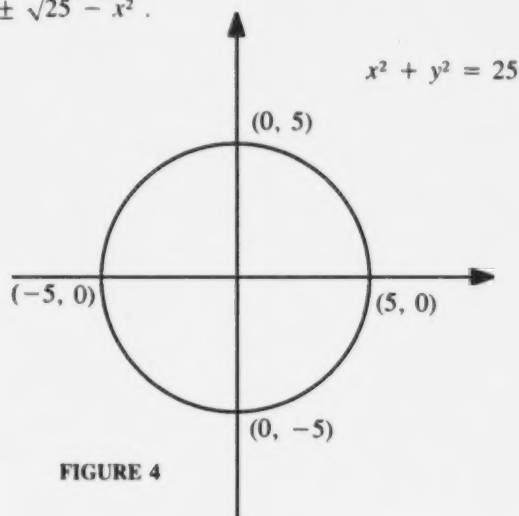


FIGURE 4

Two semicircle functions are determined; the top and bottom semicircle functions are defined respectively by:

$y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$. The top semicircle is shown in Figure 5 below.

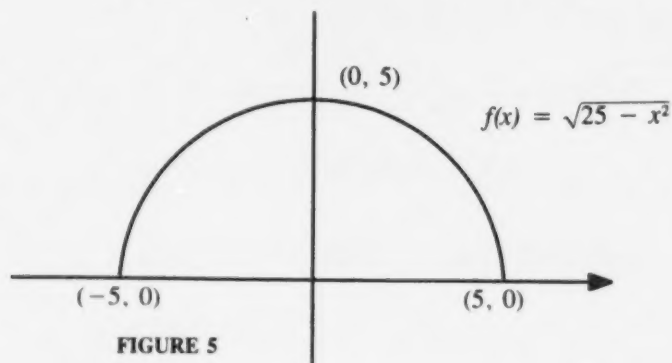


FIGURE 5

- (2) Before differentiating y or $f(x) = \sqrt{25 - x^2}$, ask students to describe characteristics of the correct derivative. Ask questions such as:

- What should happen at $(0, 5)$?
(derivative zero; horizontal tangent)
- What should happen at $(5, 0)$ and at $(-5, 0)$?
(derivative undefined; vertical tangents)
- What should happen for any x value in the first quadrant?
(derivative negative; function decreasing)
- What should happen for any x value in the second quadrant?
(derivative positive; function increasing)

- (3) Differentiate $f(x) = \sqrt{25 - x^2}$ explicitly.

$$f(x) = \sqrt{25 - x^2}$$

$$f'(x) = \frac{1}{2} (25 - x^2)^{-1/2} (-2x)$$

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}} \text{ or since } y = \sqrt{25 - x^2}$$

$$f'(x) = \frac{-x}{y}$$

- (4) Check to see that this derivative satisfies all of the conditions described in Step (2).

- (5) Differentiate $x^2 + y^2 = 25$ implicitly.

Let $y =$ some differentiable function of x .

$$\text{Then, } x^2 + [f(x)]^2 = 25$$

$$2x + 2 f(x) f'(x) = 0$$

$$x + y y' = 0$$

$$y' = \frac{-x}{y}$$

- (6) Note that the implicit result is identical to the explicit result.

- (7) Examine the derivative. Ask questions that allow students to discover that the implicit derivative gives the derivative not only for $f(x) = \sqrt{25 - x^2}$ but also for $f(x) = -\sqrt{25 - x^2}$.

(8) Have students:

- use the implicit derivative to find the slope of the tangent line to the graph of $y = -\sqrt{25 - x^2}$ at the point $(3, -4)$
- using the result of Part a, sketch in the tangent line at $(3, -4)$ (See Figure 6.)
- write the equation of the tangent line at $(3, -4)$

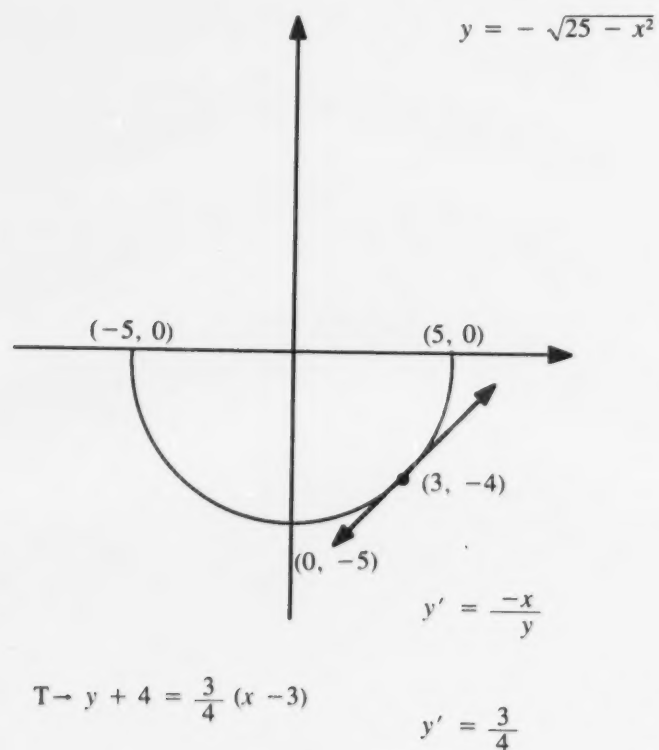


FIGURE 6

Activity 4

Topic
Differentiation

Sample Objective 9

Find the derivative of a function $f(x)$ by the definition.

Background

Students need experience in calculating derivatives using the definition in order to completely understand the definition. Simple functions may be used for this purpose to enable students to focus on the concept without being overburdened with the calculation. Have students use computer graphing technology (when available) to help them understand graphically the concept of differentiation.

Instructional Approach

Consider the function defined by $f(x) = \sqrt{2x + 6}$. (See Figure 7.)

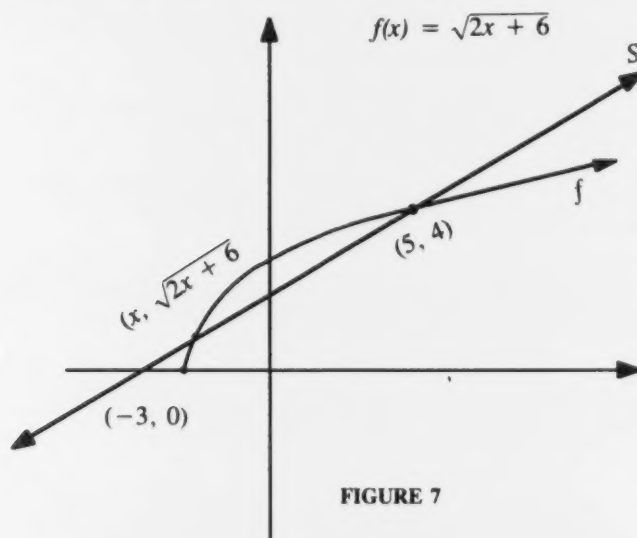


FIGURE 7

- (1) Select any specific point on the graph of f . [An integral point such as $(5, 4)$ is a good choice.]
- (2) Select a general point on the graph of f . Label the point $(x, \sqrt{2x + 6})$.
- (3) Draw the secant line connecting the two points.

- (4) Write the slope of the secant line.

$$m(x) = \frac{\sqrt{2x + 6} - 4}{x - 5}$$

- (5) Calculate

$$\lim_{x \rightarrow 5} m(x) = \lim_{x \rightarrow 5} \frac{\sqrt{2x + 6} - 4}{x - 5}$$

Use the strategy of rationalizing the numerator.

$$\begin{aligned}
 &= \lim_{x \rightarrow 5} \frac{\sqrt{2x+6}-4}{x-5} \cdot \frac{\sqrt{2x+6}+4}{\sqrt{2x+6}+4} \\
 &= \lim_{x \rightarrow 5} \frac{2x+6-16}{(x-5)(\sqrt{2x+6}+4)} \\
 &= \lim_{x \rightarrow 5} \frac{2(x-5)}{(x-5)(\sqrt{2x+6}+4)} \\
 &= \frac{1}{4}
 \end{aligned}$$

- (6) Sketch the line that contains the point $(5, 4)$ and has slope $\frac{1}{4}$.

- (7) Write the equation of the line sketched in Step (6).

$$y - 4 = \frac{1}{4} (x - 5)$$

- (8) Repeat this exercise using $(x, \sqrt{2x+6})$ and $(a, \sqrt{2a+6})$ (See Figure 8.) [In Step 5, take the limit as $x \rightarrow a$.]

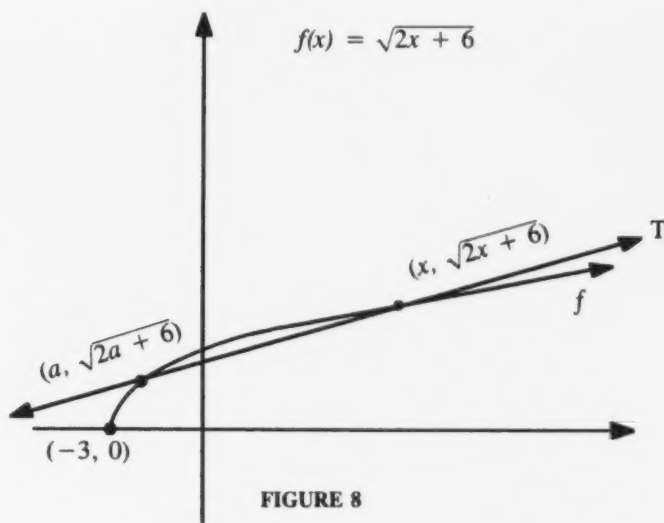


FIGURE 8

- (9) Practice and extend using functions with other characteristics such as $f(x) = \frac{4}{x-1}$

Activity 5 [Computer, Step (2)]

Topic

Area Under a Curve

Sample Objective 32

Find area under a curve using a limit of Riemann sums.

Background

Investigation of the definite integral can be begun by introducing Riemann sums. The study of Riemann sums leads to the definition of the definite integral and to an understanding of how definite integrals are used in solving physical problems. This activity reviews formulas for sums of series, examines methods of approximating area, and finally uses a limit to find the area under a curve.

Instructional Approach

- (1) Review formulas for sums of series.

- a. Evaluate the sum of the 100 terms of the series $81297 + 81495 + 81693 + \cdots + 100899$.

This arithmetic series is associated with Gauss, who before the age of 10 had devised the formula for such a sum.

Review the formula for the sum of the arithmetic series as follows:

- b. Help students discover a formula by evaluating the sum of the series A_n : $1 + 2 + 3 + 4 + \cdots + n$, followed by $1 + 2 + 3 + \cdots + (n - 1)$.
- c. Ask students if the sum of S_n : $1^2 + 2^2 + 3^2 + \cdots + n^2$ can be obtained using the same formula.

Since this is not an arithmetic or geometric series, another method is then developed, that of a table. The formula for the sum of the series A_n is first demonstrated, using partial sums:

n	A_n	$\frac{A_n}{n}$
1	1	$\frac{1}{1} = 1 = \frac{2}{2}$
2	3	$\frac{3}{2}$
3	6	$\frac{6}{3} = 2 = \frac{4}{2}$
4	10	$\frac{10}{4} = \frac{5}{2}$

$$\frac{A_n}{n} = \frac{(n + 1)}{2}$$

$$A_n = \frac{n(n + 1)}{2}$$

$$\text{and } \sum_{i=1}^n i = \frac{n(n + 1)}{2}$$

- d. Try a table of partial sums for the series S_n . Some should discover that the series A_n is related. If such does not occur, suggest that they include the column of partial sums of the series A_n .

n	S_n	A_n	$\frac{S_n}{A_n}$
1	1	1	$\frac{1}{1} = 1 = \frac{3}{3}$
2	5	3	$\frac{5}{3}$
3	14	6	$\frac{14}{6} = \frac{7}{3}$
4	30	10	$\frac{30}{10} = 3 = \frac{9}{3}$

$$\frac{S_n}{A_n} = \frac{(2n + 1)}{3}$$

$$S_n = \frac{A_n(2n + 1)}{3}$$

and so, from Part c,

$$S_n = \frac{n(n + 1)(2n + 1)}{6}$$

$$\text{and } \sum_{k=1}^n k^2 = \frac{n(n + 1)(2n + 1)}{6}$$

- (2) a. Next use a computer program to approximate the area bounded by the curve $y = x^2$, $x = 0$, $x = 2$, and the x -axis, using inscribed and circumscribed rectangles. Take the number n of subintervals to be 5, 10, 50, 100, depending upon the restrictions of the program, and have the visual representation on the computer screen demonstrate the area being calculated.
- b. Use an overhead projector to display the enclosed region and an approximation of its area using n -inscribed rectangles. (See Figure 9.)

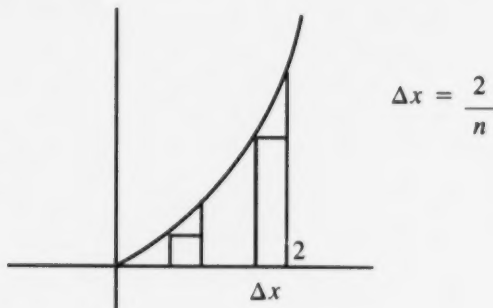


FIGURE 9

$$\begin{aligned}
 A_I &= \Delta x [f(0) + f(\Delta x) + f(2\Delta x) + f(3\Delta x) + \cdots + f((n - 1)\Delta x)] \\
 &= \Delta x [0^2 + (\Delta x)^2 + (2\Delta x)^2 + (3\Delta x)^2 + \cdots + ((n - 1)\Delta x)^2] \\
 &= (\Delta x)^3 [1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2]
 \end{aligned}$$

- c. Display the formula for the following series:

$$A_I = \frac{8}{n^3} \cdot \frac{(n-1)n(2n-1)}{6}$$

$$A_I = \frac{4}{3} \cdot \frac{(n-1)(2n-1)}{n^2}$$

- d. Finally, calculate the limit as n approaches infinity.

$$\begin{aligned} \lim_{n \rightarrow \infty} A_I &= \lim_{n \rightarrow \infty} \frac{4}{3} \cdot \frac{(n-1)(2n-1)}{n^2} \\ &= \frac{4}{3} \cdot 2 \\ &= \frac{8}{3} \end{aligned}$$

Compare the results to the integral approach and discuss the comparison.

$$\int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

Activity 6 [Calculator]

Topic

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

Sample Objective 19

Apply the theorem $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ to evaluate trigonometric limits.

Background

This activity can be used as an introduction to Objective 19, to help students discover the

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Teachers should follow this activity with many applications of the use of this limit theorem in evaluating other trigonometric limits.

Instructional Approach

- (1) Place calculator in radian mode.

- (2) Using $\theta = 1$, evaluate $\frac{\sin \theta}{\theta}$ by pressing the following keys:

[1] [sin] [÷] + [1] [=]

- (3) Repeat Step (2) for each of the following positive values of θ :

0.1, 0.01, 0.001, 0.0001, 0.00001

- (4) Repeat Step (2) for each of the following negative values of θ :

-1, -0.1, -0.01, -0.001, -0.0001, -0.00001

- (5) Conclusion: What is $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$?

Activity 7 [Computer]

Topic

Maxima and Minima

Sample Objective 27

Solve problems involving maxima and minima.

Background

Computer graphics technology enables students to observe the graphs of many functions and to determine the behavior of the graphs around maxima and minima. This activity can be used to introduce the concepts of maxima and minima. Teachers can then follow up this activity with problems involving maxima and minima.

Instructional Approach

Use the Omnifarious Plotter* in the following way:

- (1) **Plot** the graph of the function, using appropriate limits and tics.
- (2) Use the **Trace** feature to locate a relative maximum or minimum.
- (3) Use the **Zoom** feature to magnify the region near the selected point.

*Developed by Walter Dodge, New Trier High School, 385 Winnetka Avenue, Winnetka, IL 60093.

- (4) Use the **Trace** feature again to pinpoint the coordinates of the max./min.
- (5) Compare results with those found by calculus.

Activity 8 [Computer]

Topic

Area Under a Curve

Sample Objective 32

Approximate the area under a curve by summing the areas of n rectangles.

Background

Computer technology can enable students to see clearly the concept of finding area under a curve by forming rectangles under a curve and summing the areas. As the interval partition gets larger and larger, the sum of areas approaches the actual area. This activity leads nicely to the definition of the definite integral.

Instructional Approach

- (1) Write or have students write a BASIC program to allow the user to enter the function and the number of intervals. (Example follows.)

Example Program

```

10 REM THIS PROGRAM CALCULATES THE AREA UNDER A CURVE BY RECTANGLES
20 REM USING THE LEFT-HAND X OF EACH INTERVAL TO CALCULATE Y.
30 REM THE PROGRAM USES A DISK FILE TO SAVE THE FUNCTION ENTERED
40 REM AND MERGES IT BACK IN AS LINE 180.
50 REM THE PROGRAM MAY BE ENHANCED BY THE TEACHER OR STUDENT.
60 REM IT IS NOT INTENDED TO BE A SOPHISTICATED PROGRAM, ONLY
70 REM AN EXAMPLE OF HOW SUCH A PROGRAM CAN BE WRITTEN.
100 CLS
110 LOCATE 5,10: PRINT "Enter your function: F(x) = ";
120 LINE INPUT F$
130 OPEN "O", 1, "TEMP.BAS"
140 PRINT #1, "180 DEF FNF (X) = " + F$
150 CLOSE 1
160 CHAIN MERGE "TEMP.BAS", 180, ALL
180 DEF FNF (X) = 2*X 2
200 LOCATE 7, 10: INPUT "Enter a, b, and n: ",A,B,N
210 H=(B-A)/N:AREA=0
220 FOR I=0 TO N-1
230 AREA=AREA+H*FNF(A+I*H)
240 NEXT I
250 PRINT TAB (10) "The area is";AREA
260 PRINT:PRINT TAB(10):INPUT "Do you wish to change n";Y$
270 IF Y$="Y" OR Y$="y" THEN PRINT:PRINT TAB (10):INPUT "Enter new n";N:GOTO 210
    ELSE END

```

- (2) Use the Calculus Tool Kit or other software to further investigate area under a curve.

Activity 9 [Computer]

Topic

Volume by Rotation

Sample Objective 36

Find the volume of a solid formed by rotating a given region about a given line.

Background

Using computer graphics technology is an excellent way to help students understand the concepts involved in finding volume of solids of revolution. Students need to have concrete representations of the solids generated, and computer graphics makes those representations available. Computer graphics can be used to lead students into the finding of volumes by using the disc and shell methods.

Instructional Approach

If a computer laboratory is available, have the students spend a class period calculating the volume of several solids formed by revolution. A BASIC program such as that found in David L. Myers' *Precalculus Computer Activities*, Houghton Mifflin, or one developed by the teacher or a student may be used.

- (1) Begin with a semicircle, with equation $y = \sqrt{a^2 - x^2}$, and find the volume of the sphere formed by rotating the semicircle about the x -axis. The answer can be compared with the known formula.
- (2) Repeat the program for a right-circular cylinder having its axis along the x -axis. Check with the known formula.
- (3) Change the program so as to find the volume of other solids of revolution such as a cone or bullet-shaped solid.
- (4) Try at least one program that has the element perpendicular to but not connecting with the axis of revolution. For example: the region bounded by $y = x^2$ and the line $y = 4$; rotated about the x -axis. (See Figure 10.)

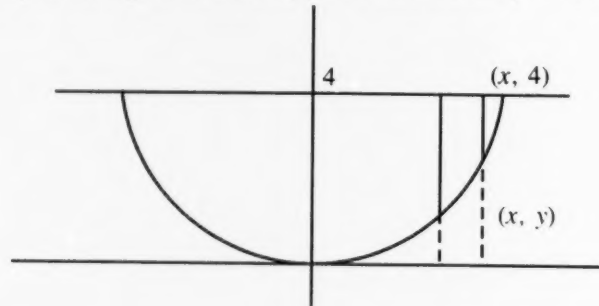


FIGURE 10

- (5) Compare the following sums:

$$\pi \sum_{x=-2}^2 (4^2 - y^2) * h \quad \text{and} \quad \pi \sum_{x=-2}^2 (4 - y)^2 * h$$

Which sum is appropriate for approximating the volume of the solid of revolution in Step (4)? Why?

Activity 10 [Graphing Calculator]

Topic

The Derivatives of the Inverse Trigonometric Functions

Sample Objective 42

Differentiate inverse trigonometric functions.

Background

The graphing calculator is an excellent tool to allow students to investigate the derivatives of the inverse trigonometric functions. Through experience of graphing several functions, students can discover characteristics that the derivatives of the inverse functions should have.

Instructional Approach

- (1) Display the graph of the inverse sine function on a graphing calculator. (See Figure 11.) Use the graph and your previous knowledge of the function to answer questions such as:

- What is the equation of this function? ($y = \arcsin x$)
- Are there other ways to write the equation?

$$(y = \sin^{-1} x; x = \sin y)$$

- What is the domain and what is the range of the function?

$$[X = (-1, 1), Y = (-\frac{\pi}{2}, \frac{\pi}{2})]$$

- Is the arcsin function an even or an odd function? How is your answer supported? (The arcsin function is odd, since $\arcsin(-x) = -\arcsin x$.)

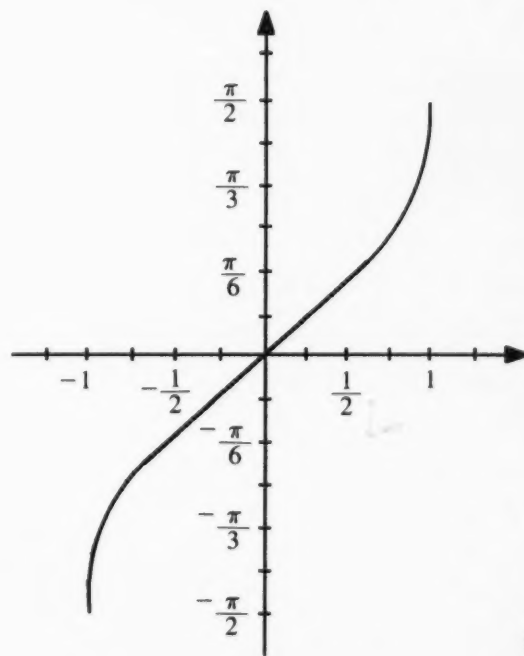


FIGURE 11

- (2) Study the graph (See Figure 11 on the preceding page.) and discuss the characteristics that the derivative of the arcsin function should have. (Guide students to make the following observations/discoveries.)
- The derivative should be positive on the interval $(-1, 1)$.
 - The derivative may be undefined at $x = -1$ and at $x = 1$. Use the zoom-in feature of the calculator to investigate this conjecture. That is, look at the graph on a small interval of the domain with -1 as its left endpoint and at a small interval of the domain with 1 as its right endpoint. (This investigation should support the original conjecture about possible vertical tangents.)
 - There is an apparent point of inflection at $x = 0$; thus, the second derivative should change sign at $x = 0$.
 - The curve is apparently concave downward on $(-1, 0)$ and concave upward on $(0, 1)$. Thus, the second derivative should be negative on $(-1, 0)$ and positive on $(0, 1)$.
- (3) Make a conjecture about the slope of the tangent line to the graph of $y = \arcsin x$ at the origin. Use the zoom feature of the calculator again. Conjecture the equation of the inflectional tangent line. (The slope of the inflectional tangent line may be 1. If the slope is 1, the equation of the inflectional tangent line would be $y = x$.)
- (4) Now, use implicit differentiation to find the derivative of the arcsin function.

If $y = \arcsin x$, then $x = \sin y$.

Recall that the domain and range accompany arcsin, and thus, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Now, let $y = f(x)$. Then,

$$x = \sin f(x)$$

$$1 = \cos f(x) f'(x)$$

$$f'(x) = \frac{1}{\cos f(x)} \text{ or } f'(x) = \frac{1}{\cos y}$$

This derivative can be expressed as a function of x . Since $x = \sin y$, by the Pythagorean identity, $\sin^2 y + \cos^2 y = 1$.

Then $\cos^2 y = \frac{1}{2} - \sin^2 y$, and since $\sin y = x$, $\cos^2 y = 1 - x^2$ or $\cos y = \pm \sqrt{1 - x^2}$.

Also, since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos y$ is positive or zero; $\cos y = \sqrt{1 - x^2}$. Now, by substitution,

$$f'(x) = \frac{1}{\sqrt{1 - x^2}}, \text{ or } D_x(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}.$$

- (5) Find $f''(x)$.

$$f''(x) = -\frac{1}{2} (1 - x^2)^{-3/2} (-2x) \text{ or } f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

- (6) Now use the first and second derivatives to confirm (or deny) the conjectures made intuitively in Step (2). Answer the following questions:

- What is the numerical sign of $f'(x)$ for all x in $(-1, 1)$?

$[f'(x) > 0 \text{ for all } x \text{ in } (-1, 1)]$

- What is $f'(-1)$ and $f'(1)$?

$[f'(-1) \text{ and } f'(1) \text{ are undefined.}]$

- What is $f''(0)$?

$[f''(0) = 0 \text{ This confirms the possible point of inflection at } x = 0 \text{ as conjectured.}]$

- What is the numerical sign of $f''(x)$ for all x in $(-1, 0)$?

$[f''(x) < 0 \text{ for all } x \text{ in } (-1, 0)]$

- What is the numerical sign of $f''(x)$ for all x in $(0, 1)$?

$[f''(x) > 0 \text{ for all } x \text{ in } (0, 1)]$ The second derivative changes sign at $x = 0$ and confirms $(0, 0)$ as a point of inflection. (Concavity conjectures are also confirmed.)

- (7) Extend this result by means of the chain rule to $y = \arcsin g(x)$. Use the graphing calculator to generate curves of the form $y = \arcsin g(x)$. Make conjectures about the derivative, and then check by means of the actual derivative. Functions chosen should include $y = \arcsin(2x)$, $y = \arcsin(10x)$, $y = \arcsin e^x$, and $y = \arcsin(\sin x)$.

- (8) A written assignment involving a similar development for $\arccos$ and/or $\arctan$ would be appropriate after the completion of this activity.

Evaluation:

Sample Tests

Evaluation is an important component in the teaching of any mathematics course and should be conducted in a carefully thought-out manner. The *Mathematics Framework, Kindergarten-Grade 12*,* page 70, outlines general procedures that should be observed in student evaluation. Calculus teachers should carefully choose or develop evaluation instruments that will best assess student knowledge and ability. The following tests are examples of effective evaluation instruments for various Calculus topics. These tests are given as samples only. Length and content may not be appropriate for specific situations. Some tests cover multiple topics and might be divided for specific uses. Some tests may be more appropriate as worksheets.

Test 1

Theory Test (Applications of the Derivative)

Read each statement carefully.

Mark T in the blank if the statement is always true; otherwise mark F and give a counter example:

- _____ 1. If $f'(c) = 0$, then $(c, f(c))$ is a point of inflection.
- _____ 2. If $f'(c) < 0$, then f is concave down at $(c, f(c))$.
- _____ 3. If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum.
- _____ 4. If $f'(c) > 0$, then the curve $f(x)$ is increasing in a neighborhood of c .
- _____ 5. If $(c, f(c))$ is a local minimum and $f'(c)$ exists, then $f'(c) = 0$.
- _____ 6. If $f'(c)$ is undefined and $f(c)$ exists, then $(c, f(c))$ may be a point of inflection.
- _____ 7. If F_1 and F_2 are functions of x on $[a, b]$, with $F_1'(x) = F_2'(x)$ on (a, b) , then $F_1(x) = F_2(x)$ all x in (a, b) .
- _____ 8. If f has a derivative at $x = c$, then f is continuous at c .

*Texas Education Agency. *Mathematics Framework, Kindergarten-Grade 12*, CU6 370 06, 1986.

- ____ 9. If f is continuous on $[a, b]$, then f has a derivative on (a, b) .
- ____ 10. If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is at least one c in (a, b) where the chord AB is parallel to the tangent line at c .
11. Let $f(x) = x^{2/3}$. Then
- ____ $f(x)$ is continuous on $[-8, 8]$.
 - ____ $f(x)$ is differentiable on $(-8, 8)$.
 - ____ $f(x)$ assumes a maximum at $x = c$, $-8 \leq c \leq 8$.
 - ____ $f(x)$ assumes a minimum at $x = c$, $-8 \leq c \leq 8$.
 - ____ $f(x)$ satisfies Rolle's Theorem on $[-8, 8]$.
 - ____ $f(x)$ satisfies the Mean Value Theorem on $[-8, 8]$.
12. If f is continuous on $[-3, 3]$, differentiable on $(-3, 3)$, with a max at $x = -1$, a min at $x = 3$, and a point of inflection at $x = 1$, then
- ____ $f'(-1) = 0$ $f''(-1) > 0$
 - ____ $f'(1) = 0$ $f''(1) = 0$
 - ____ $f'(3) = 0$ $f''(3) = 0$
 - ____ $f'(3) < 0$ $f''(-1) < 0$

Fill in the blanks with the correct response.

13. If $f(a) = -5$ and $f(b) = 3$ for some function f defined and continuous on $[a, b]$, then there is some c between a and b with $f(c) = 1$. The above is an application of the _____ Theorem.
14. If f is continuous on $[2, 8]$ and differentiable on $(2, 8)$ with $f(2) = 5$ and $f(8) = -1$, then there is some c , $2 < c < 8$, with $f'(c) =$ _____.

Write the appropriate response.

15. (a) Define $f'(x)$ _____.
- (b) Give a geometric application of $f'(x)$ _____.
- (c) Give an application to the physics of motion _____.

Test 2 (Worksheet)

Derivatives

Find the first derivative of each function.

1. $y = 6\sqrt{4+x}$
2. $y = x^2(x^2 - a^2)^{1/2}$
3. $y = (2+x)^2(1-x)^2$
4. $y = \frac{(1-2x)^2}{x}$
5. $y = \sin^4 x \cos^4 x$
6. $y = \sin(\cos x)$
7. $y = (1 + \sin^4 x)^{3/2}$

8. $y = 3 \cos 2x$
9. $y = 2 \csc (1 - 3x)$
10. $y = x^2 \cos^2 3x$
11. $y = \sec^3 x \tan^2 x$
12. $y = x + \cot x$
13. $y = \sin^{-1} 4x$
14. $y = \arctan 3x$
15. $y = \arccos \sqrt{1 - x}$
16. $y = (1 + x^2) \tan^{-1} x - x$
17. $y = (x - 1) \sqrt{2x - x^2} + \sin^{-1} (x - 1)$
18. $y = x \cos^{-1} x - \sqrt{1 - x^2}$
19. $y = 2 \sec^{-1} x - 4x^3$
20. $y = \ln(a^2 - x^2)^{3/2}$
21. $y = \frac{1 + x^2}{1 - x^2}$
22. $y = x^2 \ln x$
23. $y = \sin (\ln x)$
24. $y \ln x - x \ln y = 10$
25. $y = e^{50x} x$
26. $y = xe^{-2x}$
27. $y = e^{3x} (\cos x - 3 \sin x)$
28. $y = \arcsin e^{-x}$
29. $y = \ln (5 - e^{4x})$
30. $y = x^{-3} e^{-2x}$
31. $y = (5 - 2e^{-3x})^{1/2}$
32. $y = \ln (\sin x)$
33. $y = (x^2 + 2)e^{-2x}$
34. $y = \sqrt{1 + x} / \sqrt{1 - x}$
35. $y = \arctan (\ln x)$
36. $y = x^{x^2}$
37. $y = (\ln x)^x$
38. $y = x^{\ln x}$
39. $y = \ln (x^2 + 1) + 5^{3x}$
40. $y = 5e^{2 \ln x} + 2x^3$

Test 3

Applications of Derivatives

1. State Rolle's Theorem.
2. State the Mean Value Theorem.
3. Prove the following:
If a function $F(x)$ has a derivative that is equal to zero for all values of x in an interval (a, b) , that is, if $F'(x) = 0$ all x in (a, b) , then the function is constant throughout the interval.
4. If two functions have the same derivative for $a < x < b$, what can be said about the functions?
5. Find a number that can be chosen as the number c in the Mean Value Theorem for the given function and the interval:
 $f(x) = x + 1/x$ on $(\frac{1}{2}, 2)$
6. Show that the function $f(x) = x^5 + x^3 - x - 2$ has exactly one zero on the open interval $(1, 2)$.
7. Sand falling at the rate of $3 \text{ ft}^3/\text{min}$ forms a conical pile whose radius always equals twice the height. Find the rate at which the height is changing at the instant when the height is 10 ft.
8. A fence (straight) 90 yards long stands on a farm. The fence is to be left standing, and part or all of it is to be used in forming a rectangular pen, using an additional 260 yards of fencing for the other three sides. Find the maximum area that can be so enclosed.

9. Find horizontal and vertical asymptotes, maxima, minima, and inflection points, if any. Show each test used to identify asymptotes and critical points. Make a chart and sketch the graph.

$$f(x) = \frac{2}{(1 - x^2)}$$

Test 4

Maxima and Minima

Work four of the following advanced placement problems on maxima and minima, justifying the answer. (The year in parenthesis is the year the question appeared on a Calculus Advanced Placement Examination.)

- (1978, AB & BC) 1. Given the function f defined by $f(x) = e^{-x^2}$, find the maximum area of a rectangle that has two vertices on the x -axis and two on the graph of f .

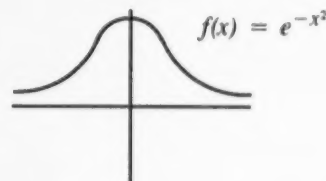


FIGURE 12

- (1980, AB) 2. A rectangle $ABCD$ with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the x -axis as shown. Find the x - and y -coordinates of C so that the area of the rectangle is a maximum.

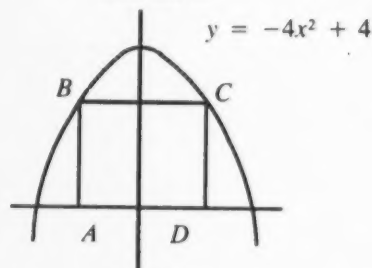


FIGURE 13

- (1957) 3. A container is to be made by fastening an open circular cylinder on a flat circular base whose radius is twice that of the cylinder. What should be the ratio of the height of the cylinder to the radius in order to produce the container of greatest capacity from a fixed amount of material?

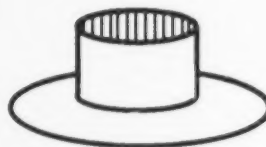


FIGURE 14

- (1966) 4. Let S denote the set of points with coordinates (x, y) such that $0 \leq x \leq 15$ and $y^2 = 150 - 10x$. Find the coordinates of the point(s) of S nearest to the origin. Also find the coordinates of the point(s) of S farthest from the origin.

- (1965) 5. A piece of wire 60'' long is cut into six sections, two of one length and four of another length. Each of the two sections having the same length is bent into the form of a circle, and the two circles are then joined by the four remaining sections to make a frame for a model of a right circular cylinder, as shown.



FIGURE 15

Find the lengths of the sections that will make the cylinder of maximum volume.

Test 5

Differentiation, Newton's Method, the Chain Rule, Continuity

- I. A. Given $xy + y^2 = 1$, find dy/dx and d^2y/dx^2 , showing the final answer in simplest form by use of the original equation and the dy/dx result.
 B. Find the equation of the line passing through the point $(2, 1)$ and normal to the curve $x^2 = 4y$.
- II. A. Given $y = 2x^3 + x$, find: 1) Δy , 2) $\Delta y_{\tan}$, 3) the difference of $\Delta y - \Delta y_{\tan}$
 B. Using differentials, obtain a reasonable approximation for $\sqrt[3]{1726}$.
- III. Use Newton's method of approximation (through x_3) in finding the root of $f(x) = 2x^3 - 3x^2 - 12x + 18 = 0$ found between $a = 1$ and $b = 2$.
- IV. Given $f(x) = u = \frac{x}{2-x}$ and $g(x) = v = \frac{x^2}{2-x}$ (f and g with natural domains)
 A. find $f \circ g$ and $g \circ f$, along with the domain of each composition
 B. find du/dx , dv/dx , dv/du , and $d(f \circ g)/dx$, all in terms of x
- V. A. Using a sketch, derive the value for $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$.
 B. Derive the formula for $\frac{d}{dx} (\sin u)$.

VI. Find dy/dx :

1. $y = u^2 - 1, x = u^2 + 1$

3. $y^3 = \sin^3 x + \cos^3 x$

2. $y = \cot^2 x$

4. $(x + 2y)^2 + 2xy^2 = 6$

VII. A. Give the definition of continuity of a function f at $x = c$.

B. For what values of x is the following function discontinuous?

$$f(x) = \frac{3x - 1}{3x^3 - 17x^2 + 21x + 9}$$

C. For $x = 2$, $f(x) = \frac{x^2 + 3x - 10}{x - 2}$. What value should be assigned to $f(2)$ to make $f(x)$ continuous at $x = 2$?

Test 6

Area Under A Curve and the Fundamental Theorems

Show all work.

1. Find the area under the curve $y = 3x^2$ from $x = 0$ to $x = 2$ by the following methods:

- the limit of areas of inscribed rectangles
- the Fundamental Theorem of Calculus
- approximation by the Trapezoidal Rule with $n = 4$. Calculate the error range.
- approximation by Simpson's Rule. Calculate the error range.

2. Evaluate the following definite integrals:

a) $\int_0^2 x^2 (x^3 + 3)^{3/2} dx$

b) $\int_0^{\pi/2} \frac{\sin x dx}{\cos^3 x}$

c) $\int_{\pi/2}^{3\pi/2} 4 \cos^2 5x dx$

d) $\int_0^{\pi} \sin^{1/2} x \cos x dx$

e) $\int_{-\pi/2}^{\pi/2} (\sin^2 2x - \cos^2 2x) dx$

f) $\int_1^3 x(3x^2 - 1)^{-1/2} dx$

3. Find $F'(x)$:

a) $F(x) = \int_2^x (t^3 + 1) dt$

b) $F(x) = \int_6^x \frac{dt}{t + 3}$

c) $F(x) = \int_a^x (3t^2 + 1)^{3/2} dt$

d) $F(x) = \int_4^x (6t - 5)^{-3/2} dt$

4. Find the area bounded by $y = 2x^3 + 1$, the x -axis, the y -axis, and the line $x = 1$.
5. Solve: $dy/dx = x^3/y^3$ if $y = 1$ when $x = 1$.
6. A particle has acceleration $a = 1$. If the initial velocity $v_0 = 3$, and $s_0 = 2$ when $t = 0$, then find the equation for:
 - a) v as a function of t
 - b) s as a function of t

Test 7

Applications of the Integral

Draw and label a figure for each problem. Show all work.

1. Find the area bounded by the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$.
2. The function $v = 3t^2 - 15t + 18$ represents the velocity v (ft/sec) of a moving body as a function of the time t (sec).
 - a) Find that portion of the time interval $0 \leq t \leq 3$ in which the velocity is positive.
 - b) Find that portion where v is negative.
 - c) Find the total distance traveled by the body from $t = 0$ to $t = 3$.
3. A train leaving a railroad station has an acceleration of $0.5 + 0.02t$ ft/sec². How far will the train move in the first 20 seconds of motion?
4. Set up but do not evaluate the integral to find the volume of the solid generated by revolving the area in the first quadrant bounded by the curve $y = 4 - x^2$ and the coordinate axes about the y -axis. (Use the shell method.)
5. Set up but do not evaluate the integral to find the volume of the solid generated by revolving the area bounded by the curve $y = 4x - x^2$ and the x -axis about the y -axis. (Use the disc method.)
6. The portion of the curve $y = \frac{(x^2 + 2)^{3/2}}{3}$ from $x = 0$ to $x = 3$, and the y -axis is rotated about the y -axis. Find:
 - a) the length of the curve from $x = 0$ to $x = 3$
 - b) the area of the surface generated
7. Find the average value of the function $y = 3x^2 - 1$ from $x = 0$ to $x = 4$.
8. Find the average value of $y = \sin^2 3x$ from $x = 0$ to $x = 3\pi/2$.

Test 8

Logarithmic and Exponential Functions

Find dy/dx .

1. $y = \ln(7 - 3x)$
2. $y = \ln \sqrt{a^2 - x^2}$
3. $y = x^{\arcsin x}$
4. $y = x^2 e^{-3x}$
5. $y = \ln(5x^2 - 2)$
6. $y = x^2 + x^4 e^{-2} \ln x$
7. $y = x^2 \ln x$
8. $y = \ln(\ln(1 - \cos x))$
9. $y = e^{2 \ln x} \sin x$
10. $y = \sin(\ln x)$

Integrate the following:

1. $\int e^{-2x} dx$
2. $\int \frac{x dx}{6x^2 - 1}$
3. $\int \frac{e^t dt}{\sqrt{1 + 5e^t}}$
4. $\int x^2 e^{x^3} dx$
5. $\int \frac{\ln y dy}{y}$
6. $\int \frac{\sec x \tan x dx}{3 + 4 \sec x}$
7. $\int \frac{\cos x dx}{2 + 3 \sin x}$
8. $\int \sin 2x e^{\cos 2x} dx$
9. $\int \frac{x^2 dx}{x^2 + 1}$
10. $\int 4e^{3 \ln x} dx$

Simplify:

1. $e^{2 \ln x}$
2. $\ln(e^{3x} - e^y)$
3. $e^{-0.5 \ln x}$
4. $e^{\ln y + 3 \ln x}$
5. $5e^{-\ln(1/x)} + 3e^{\ln y}$

Test 9

AP Calculus Sample Test

1. In the space below, state and give a general illustration of the Mean Value Theorem.
2. Show that Rolle's Theorem is satisfied by the function $f(x) = 3x^2 - x^3$ on the interval $[0, 3]$.
3. Show that the Mean Value Theorem is satisfied by f defined by $f(x) = \frac{x-1}{x}$ on $[1, 3]$.

4. Find and simplify each of the following derivatives.

a) $D_x \arcsin 4x =$

b) $D_x \arctan (x^2 + 1) =$

c) $D_x [g(x)]^6 =$

d) $D_x \sec (\sin x) =$

e) $D_x (5x^2 - 4x)^{-1/2} =$

5. Given: $f(x) = 3x^5 - 25x^3 + 60x$

Use the first derivative test to find the extrema for f . Show all work.

6. Find the points of inflection and discuss the concavity of the curve defined by:

$$f(x) = x^3 - 3x^2.$$

7. An open rectangular box is to be made from a 12'' by 12'' piece of cardboard. What is the maximum possible volume of such a box?

8. Given: $2x^2y + xy^3 = -10$

Write an equation for the tangent line to the graph defined by the given equation at the point $(2, -1)$.

Bonus:

Given: $f(x) = ax^3 + bx^2 + cx + 2$

Find values of a , b , and c so that f will contain the point $(-1, 1)$ and have a relative extremum at $(1, 2)$.

Resources



Advanced Placement Examinations

Advanced placement materials are available from the Advanced Placement Program. To order materials, write for an order form from the following address: Advanced Placement Program, P.O. Box 6670, Princeton, NJ 08541-6670. The following AP materials relate to calculus:

- *1990 AP Course Description in Mathematics*, \$5
- Set of free-response questions used in recent years, \$2
- *Teacher's Guide to AP Courses in Mathematics*, \$5
- *Grading the AP Examinations in Mathematics*, \$5
- *The 1969 Advanced Placement Examinations in Mathematics—Complete and “Unexpurgated,”* (single copy on request)
- *The Advanced Placement Program in Mathematics—Update 1975* (single copy on request)
- *The Entire 1985 AP Calculus AB Examination and Key*, \$5
- The above examination only (packets of 10), \$20
- *The Entire 1988 AP Calculus AB Examination and Key*, \$5
- The above examination only (packets of 10), \$20
- *The Entire 1985 AP Calculus BC Examination and Key*, \$5
- The above examination only (packets of 10), \$20
- *The Entire 1988 AP Calculus BC Examination and Key*, \$5
- The above examination only (packets of 10), \$20

State-Adopted Textbooks

Ellis, Robert, and Gulick, Denny. *HBJ Calculus with Analytic Geometry*. Harcourt Brace Jovanovich, Inc., 1989.

Larson, Roland E., and Hostetler, Robert P. *Calculus with Analytic Geometry*. D.C. Heath, 1986.

Thomas, George B., and Finney, Ross L. *Elements of Calculus and Analytic Geometry*. Addison-Wesley, 1989.

Supplementary Textbooks

Anton, Howard. *Calculus with Analytic Geometry*. Wiley, 1988.

Flanders, Harley, and Price, J.J. *Calculus with Analytic Geometry*. Academic Press, 1978.

Goodman, A.W., and Saff, Edward. *Analytic Geometry and the Calculus*, Fourth Ed. Macmillan, 1981.

Leithold, Louis. *Calculus with Analytic Geometry*. Harper & Row, 1986.

Munem, M.A., and Foulis, D.J. *Calculus*. D.C. Heath, 1984.

Protter, Murray, and Morey, Charles B. *Calculus with Analytic Geometry: A First Course*. Addison-Wesley, 1977.

Riddle, Douglas. *Calculus and Analytic Geometry*. Wadsworth, 1984.

Shanks, Merrill. *Calculus of Elementary Functions*. Holt, Rinehart and Winston, 1982.

Swokowski, Earl W. *Calculus with Analytic Geometry*. Prindle, Weber, and Schmidt, 1981.

Reference Materials

Ayres, Frank. *Differential and Integral Calculus*. Schaum's Outline Series, McGraw-Hill, 1980.

Bates, Jarvis, and Duyné, Van. *AP Calculus Exercises with Computer Programs*. Bates Publishing Company, 1982.

Broadwin, Judith, and Lenchner, George. *Solutions A.P. Calculus Problems Part II AB AND BC 1974-1988*. Mathematical Olympiads for Elementary Schools, SUNY/College at Old Westbury, 1988.

Eves, Howard. *An Introduction to the History of Mathematics*, Fifth Edition. New York: Holt, Rinehart and Winston, 1983.

Hockett, Shirley. *Barron's How to Prepare for the Advanced Placement Examination in Mathematics*. Barron's Educational Series, 1987.

Kenelly, John; Henry, Patricia; and Jones, Chancey. "The Advanced Placement Program in Calculus." *The Secondary School Mathematics Curriculum*. Reston, VA: The National Council of Teachers of Mathematics, 1985.

MAA Notes #6. *Toward A Lean and Lively Calculus*. The Mathematical Association of America.

MAA Notes #8. *Calculus for a New Century—A Pump, Not a Filter*. The Mathematics Association of America, 1984.

National Council of Teachers of Mathematics. *An Agenda for Action*. Reston, VA: NCTM, 1980.

_____. *Curriculum and Evaluational Standards for School Mathematics*, Reston, VA: NCTM, 1989.

National Science Board Commission on Precollege Education in Mathematics, Science, and Technology. *Educating Americans for the 21st Century*. The Commission, 1983.

Newman, James R., editor. *The World of Mathematics*. New York: Simon and Schuster, 1956.

Software

The Calculus Student's Toolkit. Addison-Wesley Publishing, Reading, MA 01867.

Computer Activities for Calculus. D.C. Heath and Co., Lexington, MA.

Derive. The Soft Warehouse, Inc., 3615 Harding Avenue, Suite 505, Honolulu, HI 96816.

Eureka: The Solver. Borland International, 4585 Scotts Valley Drive, Scotts Valley, CA 95066.

IBM Mathematics Exploration Toolkit, International Business Machines Corp., P.O. Box 1328, W. Boca Raton, FL 33429-1328.

Maple. WATCOM, 415 Phillip Street, Waterloo, Ontario, CANADA N2L3X2.

Master Grapher. Addison-Wesley Publishing, Reading, MA 01867.

MathCAD. MathSoft, Inc., 1 Kendall Square, Cambridge, MA 02139.

Mathematica. Wolfram Research, Inc., P. O. Box 6059, Champaign, IL 61821.

MicroCalc. MathCalcEduc, 1449 Covington Drive, Ann Arbor, MI 48103.

Mu Math—83. The Soft Warehouse, Inc., 3615 Harding Avenue, Suite 505, Honolulu, HI 96816.

Interactive Experiments in Calculus. Prentice-Hall, Englewood Cliffs, N.J. 07631.

Omnifarious Plotter. Walter Dodge, New Trier High School, 385 Winnetka Avenue, Winnetka, IL 60093.

True BASIC Calculus. True BASIC, Inc., 39 S. Main Street, Hanover, NH 03755.

Associations

The following associations provide information, materials, and/or conferences relating to the teaching of mathematics:

- National Council of Teachers of Mathematics (membership open to mathematics teachers)
 - publications: *Arithmetic Teacher*, *Mathematics Teacher*, *Journal for Research in Mathematics Education*, "NCTM Newsletter"
 - conferences: annual meeting, name-of-site meetings
- National Council of Supervisors of Mathematics (membership open to mathematics supervisors)
 - publication: "NCSM Newsletter"
 - conference: annual meeting
- Texas Association of Supervisors of Mathematics (membership open to Texas mathematics supervisors)
 - publication: Newsletter
 - conference: Conference for the Advancement of Mathematics Teaching (joint sponsor with the Texas Education Agency, the Texas Council of Teachers of Mathematics, and the Texas Section of the Mathematical Association of America)
- Texas Council of Teachers of Mathematics (membership open to mathematics teachers)
 - publication: *Texas Mathematics Teacher*
 - conference: Conference for the Advancement of Mathematics Teaching (joint sponsor with the Texas Education Agency, the Texas Association of Supervisors of Mathematics, and the Texas Section of the Mathematical Association of America)

ACKNOWLEDGMENTS

Texas Education Agency staff members are producing high school curriculum documents such as this one with the help of committees of educators from different geographic regions. Members of a committee from the San Antonio area were the major contributors to the development of *Guidelines for Teaching Calculus*. The committee consisted of Loyce Collenback, North East ISD; Marilyn Rindfuss, Incarnate Word College (formerly at Texas Military Institute); and Marge Valentine, Northside ISD. In addition, numerous other mathematics educators representing school districts, education service centers, and institutions of higher education across the state reviewed drafts and offered many useful suggestions.

COMPLIANCE STATEMENT

TITLE VI, CIVIL RIGHTS ACT OF 1964; THE MODIFIED COURT ORDER, CIVIL ACTION 5281, FEDERAL DISTRICT COURT, EASTERN DISTRICT OF TEXAS, TYLER DIVISION

Reviews of local education agencies pertaining to compliance with Title VI Civil Rights Act of 1964 and with specific requirements of the Modified Court Order, Civil Action No. 5281, Federal District Court, Eastern District of Texas, Tyler Division are conducted periodically by staff representatives of the Texas Education Agency. These reviews cover at least the following policies and practices:

- (1) acceptance policies on student transfers from other school districts;
- (2) operation of school bus routes or runs on a non-segregated basis;
- (3) nondiscrimination in extracurricular activities and the use of school facilities;
- (4) nondiscriminatory practices in the hiring, assigning, promoting, paying, demoting, reassigning, or dismissing of faculty and staff members who work with children;
- (5) enrollment and assignment of students without discrimination on the basis of race, color, or national origin;
- (6) nondiscriminatory practices relating to the use of a student's first language; and
- (7) evidence of published procedures for hearing complaints and grievances.

In addition to conducting reviews, the Texas Education Agency staff representatives check complaints of discrimination made by a citizen or citizens residing in a school district where it is alleged discriminatory practices have occurred or are occurring.

Where a violation of Title VI of the Civil Rights Act is found, the findings are reported to the Office for Civil Rights, U.S. Department of Education.

If there is a direct violation of the Court Order in Civil Action No. 5281 that cannot be cleared through negotiation, the sanctions required by the Court Order are applied.

TITLE VII, CIVIL RIGHTS ACT OF 1964; EXECUTIVE ORDERS 11246 AND 11375; TITLE IX, 1973 EDUCATION AMENDMENTS; REHABILITATION ACT OF 1973 AS AMENDED; 1974 AMENDMENTS TO THE WAGE-HOUR LAW EXPANDING THE AGE DISCRIMINATION IN EMPLOYMENT ACT OF 1967; AND VIETNAM ERA VETERANS READJUSTMENT ASSISTANCE ACT OF 1972 AS AMENDED IN 1974.

It is the policy of the Texas Education Agency to comply fully with the nondiscrimination provisions of all federal and state laws and regulations by assuring that no person shall be excluded from consideration for recruitment, selection, appointment, training, promotion, retention, or any other personnel action, or be denied any benefits or participation in any programs or activities which it operates on the grounds of race, religion, color, national origin, sex, handicap, age, or veteran status (except where age, sex, or handicap constitute a bona fide occupational qualification necessary to proper and efficient administration). The Texas Education Agency makes positive efforts to employ and advance in employment all protected groups.



TEXAS EDUCATION AGENCY

Austin, Texas

GE0 621 03

Spring 1990